

On reversible cascades in scale-free and Erdős-Rényi random graphs

Ching-Lueh Chang *

November 3, 2010

Abstract

Consider the following cascading process on a simple undirected graph $G(V, E)$ with diameter Δ . In round zero, a set $S \subseteq V$ of vertices, called the seeds, are active. In round $i+1$, $i \in \mathbb{N}$, a non-isolated vertex is activated if at least a $\rho \in (0, 1]$ fraction of its neighbors are active in round i ; it is deactivated otherwise. For $k \in \mathbb{N}$, let $\text{min-seed}^{(k)}(G, \rho)$ be the minimum number of seeds needed to activate all vertices in or before round k . This paper derives upper bounds on $\text{min-seed}^{(k)}(G, \rho)$. In particular, if G is connected and there exist constants $C > 0$ and $\gamma > 2$ such that the fraction of degree- k vertices in G is at most C/k^γ for all $k \in \mathbb{Z}^+$, then $\text{min-seed}^{(\Delta)}(G, \rho) = O(\lceil \rho^{\gamma-1} |V| \rceil)$. Furthermore, for $n \in \mathbb{Z}^+$, $p = \Omega((\ln(e/\rho))/(\rho n))$ and with probability $1 - \exp(-n^{\Omega(1)})$ over the Erdős-Rényi random graphs $G(n, p)$, $\text{min-seed}^{(1)}(G(n, p), \rho) = O(\rho n)$.

1 Introduction

Let $G(V, E)$ be a simple directed graph, $\rho \in (0, 1]$ and $S \subseteq V$, where each vertex of G can be in one of two states, active or inactive. The synchronous reversible cascade proceeds in rounds. In round zero, only the vertices in S , called the seeds, are active. In round $i+1$, a vertex with a positive indegree is activated (resp., deactivated) if at least (resp., less than) a ρ

*Department of Computer Science and Engineering, Yuan Ze University, Taoyuan, Taiwan. Email: clchang@saturn.yzu.edu.tw

fraction of its in-neighbors are active in round i , where $i \in \mathbb{N}$. For $k \in \mathbb{N}$, define $\text{min-seed}^{(k)}(G, \rho)$ to be the minimum number of seeds needed so that all vertices will be active in or before round k .

The synchronous reversible cascade above is the same as the local interaction game with full rationality except that the latter is defined on infinite graphs with finite degrees [65, 51]. For local interaction games, Morris [65] studies conditions allowing finitely many seeds to activate each vertex sooner or later. Blume [13], Ellison [37], Young [85, 86] and Montanari and Saberi [61] study a variant where there is another variable specifying the vertices' degree of rationality and the states are updated according to a Poisson clock. In their fully rational scenario, updating the state of a vertex v means activating or deactivating v , respectively, if at least or less than a certain fraction of v 's neighbors are active. They analyze the expected waiting time until all or most vertices enter the same state.

Consider the special cases of the synchronous reversible cascade with $\rho = 1/2$ or $\rho = 1/2 + 1/(2|V|)$. So a vertex is activated in a round if the simple or strict majority of its in-neighbors are active in the previous round; it is deactivated otherwise. Both special cases, among other similar dynamics, are suitable for modeling transient faults in majority-based fault-tolerant systems [74, 41, 42]. Call a set of seeds a k -round monopoly if it activates all vertices in or before round k , where $k \in \mathbb{N}$. Any finite-round monopoly is called a dynamic monopoly. If a set of seeds does not lead to the deactivation of active vertices in any round, then it is said to be monotone. Peleg [73] shows an $\Omega(\sqrt{|V|})$ lower bound on the minimum size of 2-round monopolies in any simple undirected graph $G(V, E)$. Bounds are known on the minimum size of monotone dynamic monopolies in planar graphs [41], toroidal meshes, torus cordales, torus serpentine [42] and simple undirected graphs [73]. Berger [10] shows the existence of constant-size dynamic monopolies in an infinite family of simple undirected graphs. Optimal bounds on the minimum size of 1-round monopolies are also derived for planar graphs, hypercubes, graphs with a given girth, graphs with diameter 2 [56] and simple undirected graphs [56, 72]. Bermond et al. [11] derive bounds on the minimum size of 1-round monopolies under the variant where a vertex v is activated or deactivated, respectively, if the majority or minority of the vertices within distance r from v are active, $r \geq 2$. Peleg [74] surveys the above and many related results.

As an important variant, the irreversible cascade on a graph forbids the deactivation of vertices. In its most general form, a vertex v is activated

in a round if at least $\phi(v)$ of its in-neighbors are active in the previous round, where $\phi(v) \in \mathbb{N}$. It is sometimes defined as an asynchronous process because the order of activating the vertices does not affect the set of vertices that will be active. With various threshold functions $\phi(\cdot)$, irreversible cascades describe the propagation of permanent faults in majority-based systems [59, 40, 41, 42], spread of diseases [36], complex propagation [20, 44, 80], socio-economic cascades [47, 83] and cascading failures in infrastructure or organizational networks [83]. Assuming irreversible cascades, bounds are derived on the minimum number of seeds activating all vertices eventually for complete trees, rings, butterflies, wrapped butterflies, cube-connected cycles, shuffle-exchange graphs, DeBruijn graphs, hypercubes [59, 41], toroidal meshes [57, 42, 76, 55, 36], torus cordales, torus serpentini [57, 42, 36], chordal rings [40], multidimensional cubes [7], complete multipartite graphs, regular graphs [36], Erdős-Rényi random graphs [23, 22, 21], undirected connected graphs [24] and directed graphs with positive indegrees [24, 1]. More bounds are derived on the minimum number of seeds activating all vertices in one round for near-regular graphs, bounded-degree graphs [72] and, under several variants, simple undirected graphs [11].

A family of undirected graphs, $\{G_n(V_n, E_n) \mid |V_n| = n\}_{n=1}^\infty$, is scale-free if there is a constant $2 < \gamma < 3$ such that for sufficiently large $k \in \mathbb{Z}^+$ and as $n \rightarrow \infty$, the fraction of vertices with degree k in G_n is proportional to $1/k^\gamma$. Directed scale-free graphs are defined similarly by using the in- or outdegrees of vertices instead. Many socio-economic, physical, biological and semantic networks are scale-free. Generative models of scale-free networks include preferential attachment models [5, 8, 6, 16, 68], copying models [18, 19, 52, 54], growth-deletion models [33, 29, 39], random-surfer models [12, 25], traffic-driven model [9], heuristically optimized trade-off model [38], hybrid models [75, 31, 71], semantic growth model [82] and random-graph models with given expected degree sequences [28, 30]. Many of these suitably describe the Webgraph [17, 35].

Let $G(V, E)$ be an undirected connected graph with diameter Δ , $\gamma > 2$ be a constant and $\rho \in (0, 1]$ such that the fraction of vertices with degree k in G is $O(1/k^\gamma)$. This paper proves $\text{min-seed}^{(\Delta)}(G, \rho) = O(\lceil \rho^{\gamma-1} |V| \rceil)$. As scale-free graphs typically have small distances between vertices [70, 32, 15, 30, 31], activating all vertices within Δ rounds may be fast. Furthermore, the $O(\lceil \rho^{\gamma-1} |V| \rceil)$ bound continues to hold even if the synchronous reversible cascade is modified to proceed asynchronously instead.

For $n \in \mathbb{Z}^+$ and $p \in [0, 1]$, the Erdős-Rényi random graph $G(n, p)$ is

a simple undirected graph with vertices $1, 2, \dots, n$ where each of the $\binom{n}{2}$ possible edges appears independently with probability p [14]. Assuming irreversible cascades, Chang and Lyuu [22, 21] consider the case where a non-isolated vertex of $G(n, p)$ is activated when at least a $\rho \in (0, 1]$ fraction of its neighbors are active. They prove that for a sufficiently large constant $\beta > 0$, $n \in \mathbb{Z}^+$, $\delta \in \{1/n, 2/n, \dots, n/n\}$, $p \geq \beta(\ln(e/\min\{\delta, \rho\}))/(\rho n)$ and with probability $1 - n^{-\Omega(1)}$ over $G(n, p)$, the minimum number of seeds needed to eventually activate at least δn vertices is $\Theta(\min\{\delta, \rho\} n)$. The hidden constant in the $\Theta(\cdot)$ notation is independent of p . For a sufficiently large constant $\lambda > 0$, $n \in \mathbb{Z}^+$, $p \geq \lambda(\ln(e/\rho))/n$ and with probability $1 - n^{-\Omega(1)}$ over $G(n, p)$, they also prove the existence of $O(\lceil \rho n \rceil)$ seeds that activate all vertices eventually.

This paper proves that for $n \in \mathbb{Z}^+$, $p = \Omega((\ln(e/\rho))/(\rho n))$ and with probability $1 - \exp(-n^{\Omega(1)})$ over $G(n, p)$, $\text{min-seed}^{(1)}(G(n, p), \rho) = O(\rho n)$. Together with Chang and Lyuu's $\Theta(\min\{\delta, \rho\} n)$ bound for irreversible cascades with an unbounded duration, our result shows that neither the reversibility of synchronous cascades nor the number of rounds played can change the asymptotically minimum number of seeds needed to activate all vertices for $p \geq \beta(\ln(e/\min\{\delta, \rho\}))/(\rho n)$ with a sufficiently large constant β — it is always $\Theta(\rho n)$. Furthermore, our $O(\rho n)$ bound continues to hold even if the synchronous reversible cascade is modified to proceed asynchronously instead.

Other related topics include periodic behavior of synchronous reversible cascades [46, 45, 77, 78, 43, 62, 63, 64], inapproximability of minimum-seed problems [26, 24, 1], majority consensus computers [69], stable sets of active vertices [2, 4, 3, 48, 53], network decontamination [60, 58], maximization of social influence [34, 79, 49, 50, 66] and percolation theory [81], among others.

This paper is organized as follows. Sec. 2 presents the notations and the preliminaries. Sec. 3 derives general bounds on the minimum number of seeds needed to activate all vertices in a number of rounds. Secs. 4–5 investigate the cases of connected scale-free and Erdős-Rényi random graphs, respectively. Sec. 6 concludes the paper.

2 Definitions

A directed graph $G(V, E)$ consists of a set V of vertices and a set $E \subseteq V \times V$ of edges. An edge $(u, v) \in E$ goes from u to v . An undirected graph is a directed

one with each edge accompanied by the edge in the opposite direction. Unless otherwise specified, all graphs in this paper are simple, i.e., self-loops are not allowed [84]. For $v \in V$, define

$$\begin{aligned} N^{\text{in}}(v) &\equiv \{u \in V \mid (u, v) \in E\}, \\ N^{\text{in}}[v] &\equiv N^{\text{in}}(v) \cup \{v\} \end{aligned}$$

to be the open and closed in-neighborhoods of v , respectively. The indegree of v is $d^{\text{in}}(v) \equiv |N^{\text{in}}(v)|$. For $A \subseteq V$, define $N^{\text{in}}(A) \equiv \cup_{a \in A} N^{\text{in}}(a)$ as the set of vertices incident on an edge coming into A . In case G is undirected, write $N^{\text{in}}(\cdot)$, $N^{\text{in}}[\cdot]$ and $d^{\text{in}}(\cdot)$ simply as $N(\cdot)$, $N[\cdot]$ and $d(\cdot)$, respectively. For an undirected graph $G(V, E)$, $U \subseteq V$ and $i \geq 0$, define $N^i[U]$ to be the set of vertices with distance less than or equal to i from at least one vertex in U . That is, $N^0[U] = U$ and $N^{i+1}[U] = N^i[U] \cup N(N^i[U])$ for $i \geq 0$. Furthermore, each vertex can be in one of two states, active or inactive.

The synchronous reversible cascade on $G(V, E)$ with seed set $S \subseteq V$ and threshold $\rho \in (0, 1]$ proceeds in rounds. In round zero, only the vertices in S , called the seeds, are active. For each $i \in \mathbb{N}$, a vertex with a positive indegree is activated or deactivated in round $i + 1$, respectively, if at least or less than a ρ fraction of its in-neighbors are active in round i . Vertices with indegree zero, instead, never change their states. For $k \in \mathbb{N}$, define $\text{Active}^{(k)}(S, G, \rho)$ to be the set of active vertices in round k . Then define

$$\text{min-seed}^{(k)}(G, \rho) \equiv \min_{W \subseteq V, \text{Active}^{(k)}(W, G, \rho) = V} |W|,$$

which is the minimum number of seeds needed to activate all vertices in round k (it is possible that a set of seeds activates all vertices in round k by doing so before round k). Clearly, once all vertices are active, they will remain active forever. Therefore, $\text{min-seed}^{(k)}(\cdot, \cdot)$ monotonically decreases as k increases.

Next, we describe an asynchronous reversible cascade. At any instant, one or more vertices may update their states. When a vertex with a positive indegree updates its state, it is activated or deactivated if at least or less than a ρ fraction of its in-neighbors are active, respectively. Instead, vertices with indegree zero never update their states. The only requirement on the synchronization of updates is that, if the state of a vertex can change at time t , then at least one vertex must change its state after time t , where $t \geq 0$. Such a synchronization is said to be progressive. In other

words, whenever changes of states are possible, an asynchronous reversible cascade cannot simply refuse them all forever. Define $\text{min-seed}^{\text{async}}(G, \rho)$ to be the minimum number of seeds that activate all vertices within a finite amount of time in every asynchronous reversible cascade with threshold ρ . So $\text{min-seed}^{\text{async}}(G, \rho) \leq s$ precisely when there exist s seeds activating all vertices within a finite amount of time no matter how the updates of states are synchronized in a progressive way. Furthermore, define $\text{min-seed}_{\text{monotone}}^{\text{async}}(G, \rho)$ to be the minimum number of seeds meeting the following criteria in every asynchronous reversible cascade with threshold ρ :

- All vertices are active after a finite amount of time;
- No active vertices are ever deactivated.

So $\text{min-seed}_{\text{monotone}}^{\text{async}}(G, \rho) \leq s$ precisely when there exist s seeds that, regardless of the (progressive) synchronizations of the reversible cascades, activate all vertices without ever deactivating any active one.

The following fact is folklore. See, e.g., [40, pp. 25, Sec. 2].

Fact 1. *Let $G(V, E)$ be a directed graph, $\rho \in (0, 1]$, $t \in \mathbb{Z}^+$ and $S \subseteq V$ satisfy*

$$\begin{aligned} S &\subseteq \text{Active}^{(1)}(S, G, \rho), \\ V &= \text{Active}^{(t)}(S, G, \rho). \end{aligned} \tag{1}$$

Then

$$\text{min-seed}_{\text{monotone}}^{\text{async}}(G, \rho) \leq |S|.$$

The following is Markov's inequality.

Fact 2. *([67, Theorem 3.2]) Let X be a random variable taking nonnegative values. Then for each $t > 0$,*

$$\Pr[X \geq t] \leq \frac{E[X]}{t}.$$

We will use the following form of Chernoff's bound [27].

Fact 3. *([67, Theorem 4.2]) Let X_1, X_2, \dots, X_k be independent random variables taking values in $\{0, 1\}$ and $p \in [0, 1]$ such that $\Pr[X_i = 1] = p$ for $1 \leq i \leq k$. Then for each $\delta \in (0, 1)$,*

$$\Pr\left[\sum_{i=1}^k X_i \leq (1 - \delta)kp\right] \leq \exp\left(-\frac{\delta^2 kp}{2}\right)$$

For $n \in \mathbb{Z}^+$ and $p \in [0, 1]$, the Erdős-Rényi random graph $G(n, p)$ is the simple undirected graph with vertices $1, 2, \dots, n$ where each of the possible $\binom{n}{2}$ edges appears independently with probability p [14]. Below is an easy consequence of Chernoff's bound.

Fact 4. ([22, Lemma 9]) *Let $n \in \mathbb{Z}^+$, $p \in [0, 1]$ and $\kappa \in \{1/n, \dots, n/n\}$. If*

$$0.99 \leq \frac{\binom{\kappa n}{2} + \kappa n(n - \kappa n)}{\kappa n^2} \leq 1,$$

then

$$\Pr \left[\left| \left\{ v \in [n] \mid d(v) \leq \frac{pn}{2} \right\} \right| \geq \kappa n \right] \leq \binom{n}{\kappa n} \exp \left(-\frac{\kappa p n^2}{9} \right),$$

where the probability is taken over the Erdős-Rényi random graphs $G(n, p)$.

The irreversible cascade is the modification of our synchronous reversible cascade that prohibits the deactivation of vertices. It can also be defined as an asynchronous process without affecting the set of vertices that will be active [40, pp. 25, Sec. 2]. Assuming irreversible cascades, Chang and Lyuu [22, 21] prove the following bound on activating vertices of Erdős-Rényi random graphs.

Fact 5. ([22, 21]) *Assume irreversible cascades. For a sufficiently large constant $\beta > 0$, $n \in \mathbb{Z}^+$, $\rho \in (0, 1]$, $\delta \in \{1/n, 2/n, \dots, n/n\}$, $p \geq \beta(\ln(e/\min\{\delta, \rho\}))/(\rho n)$ and with probability $1 - n^{-\Omega(1)}$ over $G(n, p)$, the minimum number of seeds needed to eventually activate at least δn vertices of $G(n, p)$ is $\Theta(\min\{\delta, \rho\} n)$. The hidden constant in the $\Theta(\cdot)$ notation is independent of p .*

3 General bounds

We begin with several lemmas that will be useful for analyzing reversible cascades on scale-free graphs.

Lemma 6. *Let $G(V, E)$ be an undirected connected graph with diameter Δ , $|V| \geq 2$ and $\{v \in V \mid d(v) > 1/\rho\} \neq \emptyset$. Then*

$$\text{min-seed}^{(\Delta)}(G, \rho) \leq \sum_{v \in V, d(v) > 1/\rho} (\lceil \rho d(v) \rceil + 1). \quad (2)$$

Proof. For each $v \in V$, pick a set $B(v) \subseteq N(v)$ of size $\lceil \rho d(v) \rceil$. Let

$$X = \left\{ v \in V \mid d(v) > \frac{1}{\rho} \right\}. \quad (3)$$

Then take

$$S = \bigcup_{v \in X} (B(v) \cup \{v\})$$

as the set of seeds. Clearly, $S \neq \emptyset$. As the righthand side of Eq. (2) is an upper bound on $|S|$, it suffices to establish $\text{Active}^{(\Delta)}(S, G, \rho) = V$.

For any $u \in S$, we have

$$|N(u) \cap S| \geq \lceil \rho d(u) \rceil \quad (4)$$

by the following arguments:

- If $u \in X$, then $B(u) \subseteq S$, implying Eq. (4).
- Otherwise, $u \in B(v)$ for some $v \in X$. Therefore, $v \in N(u) \cap S$, implying Eq. (4) because $d(u) \leq 1/\rho$.

By Eq. (4), if all vertices in S are active in a round, then they will remain to be active in the next round. Therefore, as the vertices in S are active in round zero,

$$S \subseteq \bigcap_{k \geq 0} \text{Active}^{(k)}(S, G, \rho). \quad (5)$$

For each $i \geq 0$, denote by $P(i)$ the relation

$$N^i[S] \subseteq \bigcap_{k \geq i} \text{Active}^{(k)}(S, G, \rho). \quad (6)$$

By construction, every $w \in V \setminus S$ satisfies $d(w) \leq 1/\rho$ and thus

$$\lceil \rho d(w) \rceil = 1. \quad (7)$$

For $i \in \mathbb{N}$, $P(i)$ implies $P(i+1)$ by the following arguments:

- Every $w \in N^{i+1}[S] \setminus N^i[S]$ has a neighbor in $N^i[S]$. So by $P(i)$, w has at least one active neighbor in rounds $i, i+1, \dots$. Hence by Eq. (7) and the definition of the synchronous reversible cascade, w is active in rounds $i+1, i+2, \dots$.

- By $P(i)$, all vertices in $N^i[S]$ are active in rounds $i + 1, i + 2, \dots$

As Eq. (5) is precisely $P(0)$, $P(i)$ holds for all $i \in \mathbb{N}$ by mathematical induction. Finally, $P(\Delta)$ gives $\text{Active}^{(\Delta)}(S, G, \rho) = V$. \square

The following lemma reduces the bound of Lemma 6 to 2 if all degrees are less than or equal to ρ .

Lemma 7. *Let $G(V, E)$ be an undirected connected graph with diameter Δ , $|V| \geq 2$ and $\{v \in V \mid d(v) > 1/\rho\} = \emptyset$. Then*

$$\text{min-seed}^{(\Delta)}(G, \rho) \leq 2. \quad (8)$$

Proof. Modify the proof of Lemma 6 by taking X to contain a single vertex in Eq. (3) and replacing the reference to “Eq. (2)” by “Eq. (8).” The rest of the proof follows, word for word. \square

Lemma 8. *Let $G(V, E)$ be an undirected connected graph with diameter Δ and $|V| \geq 2$. Then*

$$\text{min-seed}^{(\Delta)}(G, \rho) \leq 2 + \sum_{v \in V, d(v) > 1/\rho} (\lceil \rho d(v) \rceil + 1).$$

Proof. Immediate from Lemmas 6–7. \square

Corollary 9. *Let $G(V, E)$ be an undirected connected graph with diameter Δ and $|V| \geq 2$. Then*

$$\text{min-seed}_{\text{monotone}}^{\text{async}}(G, \rho) \leq 2 + \sum_{v \in V, d(v) > 1/\rho} (\lceil \rho d(v) \rceil + 1).$$

Proof. Eq. (4) holds in our proofs of Lemmas 6–7, and it implies Eq. (1). Hence the corollary follows from Fact 1 and the proofs of Lemmas 6–7. \square

Next, we analyze the following picking of seeds in a directed graph $G(V, E)$: First, choose each vertex as a seed independently with some probability. Second, choose the vertices that cannot be activated in round 1 also as seeds. The expected number of chosen seeds serves as an upper bound on $\text{min-seed}^{(1)}(G, \rho)$. Chang and Lyuu [23, Theorem 9] use a similar technique to analyze irreversible cascades. Their results are improved by Ackerman, Ben-Zwi and Wolfvitz [1, Sec. 3].

Lemma 10. For a directed graph $G(V, E)$, $\rho \in (0, 1]$ and $C > 1$,

$$\text{min-seed}^{(1)}(G, \rho) \leq O(C\rho |V|) + \sum_{v \in V} (d^{\text{in}}(v) + 1) \exp(-3C\rho d^{\text{in}}(v)). \quad (9)$$

Proof. Assume $\rho < 1/(8C)$ for, otherwise, Eq. (9) holds trivially. Let $S \subseteq V$ contain each vertex independently with probability $8C\rho$ and

$$A = \{v \in V \mid |N^{\text{in}}(v) \cap S| \leq \rho d^{\text{in}}(v)\}.$$

By construction, $V \setminus A \subseteq \text{Active}^{(1)}(S, G, \rho)$. Clearly, $A \subseteq \text{Active}^{(1)}(N^{\text{in}}(A) \cup A, G, \rho)$. Consequently,

$$\text{Active}^{(1)}(S \cup N^{\text{in}}(A) \cup A, G, \rho) = V. \quad (10)$$

Clearly,

$$E[|S|] = 8C\rho |V|, \quad (11)$$

$$|N^{\text{in}}(A) \cup A| \leq \sum_{v \in A} (d^{\text{in}}(v) + 1). \quad (12)$$

By Chernoff's bound (Fact 3),

$$\Pr[v \in A] \leq \exp(-3C\rho d^{\text{in}}(v)) \quad (13)$$

for all $v \in V$. By Eqs. (12)–(13) and the linearity of expectation,

$$E[|N^{\text{in}}(A) \cup A|] \leq \sum_{v \in V} (d^{\text{in}}(v) + 1) \exp(-3C\rho d^{\text{in}}(v)).$$

This and Eqs. (10)–(11) complete the proof because there must exist a realization of $S \cup N^{\text{in}}(A) \cup A$ with size less than or equal to its expected value. \square

Lemma 11. For a directed graph $G(V, E)$, $\rho \in (0, 1]$ and $C > 1$,

$$\text{min-seed}^{(1)}(G, \rho) \leq O(C\rho |V|) + \sum_{v \in V, d^{\text{in}}(v) < (1/(C\rho)) \ln(e/\rho)} (d^{\text{in}}(v) + 1) \exp(-3C\rho d^{\text{in}}(v)).$$

Proof. By elementary calculus,

$$\max_{x \geq (1/(C\rho)) \ln(e/\rho)} (x + 1) \exp(-3C\rho x) = O(\rho).$$

Therefore, in the summation of Eq. (9), vertices with indegrees at least $(1/(C\rho)) \ln(e/\rho)$ contribute $O(\rho |V|)$ in total. \square

Lemma 12. For a directed graph $G(V, E)$, $\rho \in (0, 1]$ and $C > 1$,

$$\text{min-seed}^{(1)}(G, \rho) \leq O(C\rho |V|) + \left| \left\{ v \in V \mid d^{\text{in}}(v) < \frac{1}{C\rho} \ln \frac{e}{\rho} \right\} \right| \cdot \left(\frac{1}{C\rho} \left(\ln \frac{e}{\rho} \right) + 1 \right).$$

Proof. Invoke Lemma 11 and observe that $\exp(-3C\rho d^{\text{in}}(v)) \leq 1$ for all $v \in V$. \square

Corollary 13. Let $G(V, E)$ be a directed graph $\rho \in (0, 1]$ and $C > 1$. If every vertex of G has indegree $\Omega((1/\rho) \ln(e/\rho))$, then

$$\text{min-seed}^{(1)}(G, \rho) = O(\rho |V|).$$

Proof. Invoke Lemma 12 with a sufficiently large constant C . \square

Corollary 13 holds, e.g., for $\Omega((1/\rho) \ln(e/\rho))$ -regular graphs.

4 Bounds for connected scale-free graphs

Let $G(V, E)$ be a connected scale-free graph with diameter Δ and $2 < \gamma < 3$ be a constant such that G has an $O(1/k^\gamma)$ fraction of degree- k vertices, $k \in \mathbb{Z}^+$. This section shows that $\text{min-seed}^{(\Delta)}(G, \rho) = O(\lceil \rho^{\gamma-1} |V| \rceil)$, which suggests rapid reversible cascades because scale-free graphs typically have small diameters or average vertex-vertex distances [70, 32, 15, 30, 31]. In the case of asynchronous reversible cascades, $\text{min-seed}_{\text{monotone}}^{\text{async}}(G, \rho) = O(\lceil \rho^{\gamma-1} |V| \rceil)$. Compared with Fact 5, therefore, activating all vertices requires fewer seeds on connected scale-free graphs than on Erdős-Rényi random graphs $G(n, p)$ for $\omega(1/n) \leq \rho \leq o(1)$ and $p = \Omega((\ln(e/\rho))/(\rho n))$ with a sufficiently large hidden constant in the $\Omega(\cdot)$ notation. This holds even if we allow deactivations and require the cascades to succeed regardless of the (progressive) synchronizations in connected scale-free graphs but not in Erdős-Rényi random graphs.

Theorem 14. Let $G(V, E)$ be an undirected connected graph with diameter Δ and $\rho \in (0, 1]$. If $C > 0$ and $\gamma > 2$ are constants with

$$\frac{|\{v \in V \mid d(v) = k\}|}{|V|} \leq \frac{C}{k^\gamma} \tag{14}$$

for all $k \in \mathbb{Z}^+$, then

$$\text{min-seed}^{(\Delta)}(G, \rho) = O(\lceil \rho^{\gamma-1} |V| \rceil). \tag{15}$$

Proof. We may assume without loss of generality that $\rho < 0.1$ for, otherwise, Eq. (15) holds trivially (note that γ is a constant). By Lemma 8,

$$\text{min-seed}^{(\Delta)}(G, \rho) = O \left(2 + \sum_{v \in V, d(v) > 1/\rho} \rho d(v) \right). \quad (16)$$

Now,

$$\begin{aligned} & \sum_{v \in V, d(v) > 1/\rho} d(v) \\ \stackrel{\text{Eq. (14)}}{\leq} & \sum_{k \in \mathbb{N}, k > 1/\rho} k \cdot \frac{C |V|}{k^\gamma} \\ \leq & \int_{1/\rho-1}^{\infty} \frac{C |V|}{x^{\gamma-1}} dx \\ = & O(\rho^{\gamma-2} |V|), \end{aligned}$$

where the second inequality follows from elementary calculus and the $O(\cdot)$ notation hides constants dependent on C and γ . This and Eq. (16) complete the proof. \square

Corollary 15. *Let $G(V, E)$ be an undirected connected graph with diameter Δ and $\rho \in (0, 1]$. If $C > 0$ and $\gamma > 2$ are constants satisfying Eq. (14) for all $k \in \mathbb{Z}^+$, then*

$$\text{min-seed}_{\text{monotone}}^{\text{async}}(G, \rho) = O(\lceil \rho^{\gamma-1} |V| \rceil). \quad (17)$$

Proof. In the proof of Theorem 14, invoke Corollary 9 instead of Lemma 8. \square

5 Bounds for Erdős-Rényi random graphs

Fact 5 does not give that

$$\begin{aligned} \text{min-seed}^{(1)}(G(n, p), \rho) &= O(\rho n), \\ \text{min-seed}_{\text{monotone}}^{\text{async}}(G(n, p), \rho) &= O(\rho n) \end{aligned}$$

because of possible deactivations, the requirement of activating all vertices in only one round and the arbitrary (yet progressive) synchronizations of the cascades. Still, this section proves both equations for $p = \Omega((\ln(e/\rho))/(\rho n))$.

Lemma 16. *Let $n \in \mathbb{Z}^+$, $\rho \in [1/n^{1/3}, 1]$, $C > 1$ and $p \geq (100/(C\rho n)) \ln(e/\rho)$. Then with probability $1 - \exp(-\Omega((\rho^2 n/C) \ln(e/\rho)))$ over the Erdős-Rényi random graphs $G(n, p)$,*

$$\text{min-seed}^{(1)}(G(n, p), \rho) = O(C\rho n). \quad (18)$$

Proof. Assume $\rho < 1/(10C)$ for, otherwise, Eq. (18) holds trivially. By Fact 4,

$$\Pr \left[\left| \left\{ v \in V \mid d(v) < \frac{1}{C\rho} \ln \frac{e}{\rho} \right\} \right| \geq \lfloor \rho^3 n \rfloor \right] \leq \exp \left(-\Omega \left(\frac{\rho^2 n}{C} \ln \frac{e}{\rho} \right) \right).$$

This and Lemma 12 complete the proof. \square

Lemma 17. *Let $n \in \mathbb{Z}^+$, $\rho \in (0, 1]$, $C > 1$ and $p \geq (100/(C\rho n)) \ln(e/\rho)$. Then with probability $1 - (1/\rho^3) \exp(-\Omega((1/(C\rho)) \ln(e/\rho)))$ over the Erdős-Rényi random graphs $G(n, p)$,*

$$\text{min-seed}^{(1)}(G(n, p), \rho) = O(C\rho n).$$

Proof. By Chernoff's bound (Fact 3),

$$\Pr \left[d(v) < \frac{1}{C\rho} \ln \frac{e}{\rho} \right] \leq \exp \left(-\Omega \left(\frac{1}{C\rho} \ln \frac{e}{\rho} \right) \right),$$

$v \in \{1, 2, \dots, n\}$. So by the linearity of expectation,

$$E \left[\left| \left\{ v \in \{1, 2, \dots, n\} \mid d(v) < \frac{1}{C\rho} \ln \frac{e}{\rho} \right\} \right| \right] \leq n \exp \left(-\Omega \left(\frac{1}{C\rho} \ln \frac{e}{\rho} \right) \right),$$

implying

$$\Pr \left[\left| \left\{ v \in \{1, 2, \dots, n\} \mid d(v) < \frac{1}{C\rho} \ln \frac{e}{\rho} \right\} \right| \geq \rho^3 n \right] \leq \frac{1}{\rho^3} \exp \left(-\Omega \left(\frac{1}{C\rho} \ln \frac{e}{\rho} \right) \right)$$

by Markov's inequality (Fact 2). This and Lemma 12 complete the proof. \square

Theorem 18. *Let $n \in \mathbb{Z}^+$, $\rho \in (0, 1]$ and $p = \Omega((\ln(e/\rho))/(\rho n))$. Then with probability $1 - \exp(-n^{\Omega(1)})$ over the Erdős-Rényi random graphs $G(n, p)$,*

$$\text{min-seed}^{(1)}(G(n, p), \rho) = O(\rho n).$$

Proof. For a sufficiently large constant C , invoke Lemma 16 if $\rho \geq 1/n^{1/3}$ and Lemma 17 otherwise. \square

By Fact 5 and Theorem 18, for $p = \Omega((\ln(e/\rho))/(\rho n))$ with a sufficiently large hidden constant in the $\Omega(\cdot)$ notation,

$$\text{min-seed}^{(k)}(G(n, p), \rho) = \Theta(\rho n)$$

with probability $1 - n^{-\Omega(1)}$ for all $k \in \mathbb{Z}^+$. This is asymptotically the same as the bound in Fact 5 for irreversible cascades.

Corollary 19. *Let $n \in \mathbb{Z}^+$, $\rho \in (0, 1]$ and $p = \Omega((\ln(e/\rho))/(\rho n))$. Then with probability $1 - \exp(-n^{\Omega(1)})$ over the Erdős-Rényi random graphs $G(n, p)$,*

$$\text{min-seed}_{\text{monotone}}^{\text{async}}(G(n, p), \rho) = O(\rho n).$$

Proof. By Theorem 18, there exists a set S of $O(\rho n)$ seeds with $\text{Active}^{(1)}(S, G, \rho) = V$. Hence the theorem follows from Fact 1. \square

6 Conclusions

Reversible cascades are central in local interaction games with full rationality [65, 51] and the propagation of transient faults in majority-based systems [74, 41, 42]. We investigated them in connected scale-free and Erdős-Rényi random graphs. Suppose $\omega(1/n) \leq \rho \leq o(1)$ and $p = \Omega((\ln(e/\rho))/(\rho n))$ with a sufficiently large hidden constant in the $\Omega(\cdot)$ notation. Theorems 14, 18 and Fact 5 show that activating all vertices is easier (in terms of the number of seeds deployed) on connected scale-free graphs than on Erdős-Rényi random graphs $G(n, p)$. However, an asymptotically smallest set of seeds that activate all vertices of $G(n, p)$ (which has size $\Theta(\rho n)$ by Fact 5 and Theorem 18) can do so in one round by Theorem 18, whereas Theorem 14 does not show seeds activating all vertices of connected scale-free graphs within $O(1)$ rounds. It thus remains to further investigate the tradeoff between the number of seeds and the round complexity for activating all vertices in scale-free graphs. Such tradeoffs are studied for many graphs by Flocchini et al. [41] because of their theoretical and practical importance. Another interesting direction is to refine the $O(\lceil \rho^{\gamma-1} |V| \rceil)$ bound in Theorem 14 for many important scale-free graphs such as those mentioned in Sec. 1.

References

- [1] E. Ackerman, O. Ben-Zwi, and G. Wolfowitz. Combinatorial model and bounds for target set selection. *Theoretical Computer Science*, 2010. Forthcoming. DOI: 10.1016/j.tcs.2010.08.021.
- [2] Z. Agur. Resilience and variability in pathogens and hosts. *IMA Journal on Mathematical Medicine and Biology*, 4(4):295–307, 1987.
- [3] Z. Agur. Fixed points of majority rule cellular automata with application to plasticity and precision of the immune system. *Complex Systems*, 5(3):351–357, 1991.
- [4] Z. Agur, A. S. Fraenkel, and S. T. Klein. The number of fixed points of the majority rule. *Discrete Mathematics*, 70(3):295–302, 1988.
- [5] W. Aiello, F. Chung, and L. Lu. Random evolution in massive graphs. In J. Abello, P. M. Pardalos, and M. G. C. Resende, editors, *Handbook of Massive Data Sets*, pages 97–122. Kluwer Academic Publishers, 2002.
- [6] R. Albert and A.-L. Barabási. Statistical mechanics of complex networks. *Reviews of Modern Physics*, 74(1):47–97, 2002.
- [7] J. Balogh and G. Pete. Random disease on the square grid. *Random Structures & Algorithms*, 13(3–4):409–422, 1998.
- [8] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, 1999.
- [9] A. Barrat, M. Barthélemy, and A. Vespignani. Traffic-driven model of the World Wide Web graph. In *Proceedings of the 3rd International Workshop on Algorithms and Models for the Web-Graph*, pages 56–67, 2004.
- [10] E. Berger. Dynamic monopolies of constant size. *Journal of Combinatorial Theory Series B*, 83(2):191–200, 2001.
- [11] J.-C. Bermond, J. Bond, D. Peleg, and S. Perennes. The power of small coalitions in graphs. *Discrete Applied Mathematics*, 127(3):399–414, 2003.

- [12] A. Blum, T.-H. H. Chan, and M. R. Rwebangira. A random-surfer web-graph model. In *Proceedings of the 8th Workshop on Algorithm Engineering and Experiments*, pages 238–246, 2006.
- [13] L. E. Blume. The statistical mechanics of strategic interaction. *Games and Economic Behavior*, 5(3):387–424, 1993.
- [14] B. Bollobás. *Random Graphs*. Cambridge University Press, 2nd edition, 2001.
- [15] B. Bollobás and O. Riordan. The diameter of a scale-free random graph. *Combinatorica*, 24(1):5–34, 2004.
- [16] B. Bollobás, O. Riordan, J. Spencer, and G. Tusnády. The degree sequence of a scale-free random graph process. *Random Structures and Algorithms*, 18(3):279–290, 2001.
- [17] A. Bonato. A survey of models of the Web graph. In A López-Ortiz and A. Hamel, editors, *Proceedings of the 1st Workshop on Combinatorial and Algorithmic Aspects of Networking*, pages 159–172. Springer-Verlag Berlin Heidelberg, 2004.
- [18] A. Bonato and J. Janssen. Infinite limits of copying models of the Web graph. *Internet Mathematics*, 1(2):193–213, 2004.
- [19] A. Bonato and J. Janssen. Limits and power laws of models for the Web graph and other networked information spaces. In A López-Ortiz and A. Hamel, editors, *Proceedings of the 1st Workshop on Combinatorial and Algorithmic Aspects of Networking*, pages 42–48. Springer-Verlag Berlin Heidelberg, 2004.
- [20] D. Centola, V. M. Eguíluz, and M. W. Macy. Cascade dynamics of complex propagation. *Physica A: Statistical Mechanics and its Applications*, 374(1):449–456, 2007.
- [21] C.-L. Chang. *Analysis of Threshold-Based Processes of Cascading Activation in Graphs*. PhD thesis, National Taiwan University, 2010. Advisor: Prof. Yuh-Dauh Lyuu.
- [22] C.-L. Chang and Y.-D. Lyuu. Spreading of messages in random graphs. *Theory of Computing Systems*. Forthcoming. DOI: 10.1007/s00224-010-9258-7.

- [23] C.-L. Chang and Y.-D. Lyuu. Spreading messages. *Theoretical Computer Science*, 410(27–29):2714–2724, 2009.
- [24] C.-L. Chang and Y.-D. Lyuu. Bounding the number of tolerable faults in majority-based systems. In T. Calamoneri and J. Díaz, editors, *Proceedings of the 7th International Conference on Algorithms and Complexity*, pages 109–119, Rome, Italy, 2010. Springer-Verlag Berlin Heidelberg.
- [25] P. Chebolu and P. Melsted. PageRank and the random surfer model. In *Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1010–1018, 2008.
- [26] N. Chen. On the approximability of influence in social networks. In *Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1029–1037, 2008.
- [27] H. Chernoff. A measure of the asymptotic efficiency of tests of a hypothesis based on the sum of observations. *Annals of Mathematical Statistics*, 23(4):493–507, 1952.
- [28] F. Chung and L. Lu. Connected components in random graphs with given expected degree sequences. *Annals of Combinatorics*, 6(2):125–145, 2002.
- [29] F. Chung and L. Lu. Coupling online and offline analyses for random power law graphs. *Internet Mathematics*, 1(4):409–461, 2003.
- [30] F. Chung and L. Lu. The average distance in a random graph with given expected degrees. *Internet Mathematics*, 1(1):91–114, 2004.
- [31] F. Chung and L. Lu. The small world phenomenon in hybrid power law graphs. In E. Ben-Naim, H. Frauenfelder, and Z. Toroczkai, editors, *Complex Networks*, pages 89–104. Springer-Verlag Berlin Heidelberg, 2004.
- [32] R. Cohen and S. Havlin. Scale-free networks are ultrasmall. *Physical Review Letters*, 90(5):058701, 2003.
- [33] C. Cooper, A. M. Frieze, and J. Vera. Random deletion in a scale-free random graph process. *Internet Mathematics*, 1(4):463–483, 2003.

- [34] P. Domingos and M. Richardson. Mining the network value of customers. In *Proceedings of the 7th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 57–66, 2001.
- [35] D. Donato, L. Laura, S. Leonardi, and S. Millozzi. Simulating the Webgraph: A comparative analysis of models. *Computing in Science and Engineering*, 6(6):84–89, 2004.
- [36] P. A. Dreyer and F. S. Roberts. Irreversible k -threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion. *Discrete Applied Mathematics*, 157(7):1615–1627, 2009.
- [37] G. Ellison. Learning, local interaction, and coordination. *Econometrica*, 61(5):1047–1071, 1993.
- [38] A. Fabrikant, E. Koutsoupias, and C. H. Papadimitriou. Heuristically optimized trade-offs: A new paradigm for power laws in the Internet. In *Proceedings of the 29th International Colloquium on Automata, Languages and Programming*, pages 110–122, 2002.
- [39] A. D. Flaxman, A. M. Frieze, and J. Vera. Adversarial deletion in a scale-free random graph process. *Combinatorics, Probability and Computing*, 16(2):261–270, 2007.
- [40] P. Flocchini, F. Geurts, and N. Santoro. Optimal irreversible dynamos in chordal rings. *Discrete Applied Mathematics*, 113(1):23–42, 2001.
- [41] P. Flocchini, R. Kráľovič, P. Ružička, A. Roncato, and N. Santoro. On time versus size for monotone dynamic monopolies in regular topologies. *Journal of Discrete Algorithms*, 1(2):129–150, 2003.
- [42] P. Flocchini, E. Lodi, F. Luccio, L. Pagli, and N. Santoro. Dynamic monopolies in tori. *Discrete Applied Mathematics*, 137(2):197–212, 2004.
- [43] Y. Ginosar and R. Holzman. The majority action on infinite graphs: Strings and puppets. *Discrete Mathematics*, 215(1–3):59–71, 2000.
- [44] J. P. Gleeson and D. J. Cahalane. Seed size strongly affects cascades on random networks. *Physical Review E*, 75(5):056103, 2007.
- [45] E. Goles and J. Olivos. Periodic behavior of generalized threshold functions. *Discrete Mathematics*, 30(2):187–189, 1980.

- [46] E. Goles-Chacc, F. Fogelman-Soulie, and D. Pellegrin. Decreasing energy functions as a tool for studying threshold networks. *Discrete Applied Mathematics*, 12(3):261–277, 1985.
- [47] M. Granovetter. Threshold models of collective behavior. *The American Journal of Sociology*, 83(6):1420–1443, 1978.
- [48] A. Granville. On a paper by Agur, Fraenkel and Klein. *Discrete Mathematics*, 94(2):147–151, 1991.
- [49] D. Kempe, J. Kleinberg, and É. Tardos. Maximizing the spread of influence through a social network. In *Proceedings of the 9th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 137–146, 2003.
- [50] D. Kempe, J. Kleinberg, and É. Tardos. Influential nodes in a diffusion model for social networks. In *Proceedings of the 32nd International Colloquium on Automata, Languages and Programming*, pages 1127–1138, 2005.
- [51] J. Kleinberg. Cascading behavior in networks: Algorithmic and economic issues. In N. Nisan, T. Roughgarden, É. Tardos, and V. Vazirani, editors, *Algorithmic Game Theory*. Cambridge University Press, 2007.
- [52] J. Kleinberg, R. Kumar, P. Raghavan, S. Rajagopalan, and A. Tomkins. The Web as a graph: Measurements, models, and methods. In *Proceedings of the 5th Annual International Computing and Combinatorics Conference*, pages 1–17, 1999.
- [53] R. Kráľovič. On majority voting games in trees. In *Proceedings of the 28th Conference on Current Trends in Theory and Practice of Informatics*, pages 282–291, 2001.
- [54] R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins, and E. Upfal. Stochastic models for the Web graph. In *Proceedings of the 41st Annual Symposium on Foundations of Computer Science*, pages 57–65, 2000.
- [55] J. Kynčl, B. Lidický, and T. Vyskočil. Irreversible 2-conversion set is NP-complete. Technical Report KAM-DIMATIA Series 2009-933, Department of Applied Mathematics, Charles University, Prague, Czech Republic, 2009.

- [56] N. Linial, D. Peleg, Y. Rabinovich, and M. Saks. Sphere packing and local majorities in graphs. In *Proceedings of the 2nd Israel Symposium on Theory of Computing Systems*, pages 141–149, 1993.
- [57] F. Luccio. Almost exact minimum feedback vertex set in meshes and butterflies. *Information Processing Letters*, 66(2):59–64, 1998.
- [58] F. Luccio and L. Pagli. Web marshals fighting curly link farms. In *Proceedings of the 4th International Conference on Fun with Algorithms*, pages 240–248, 2007.
- [59] F. Luccio, L. Pagli, and H. Sanossian. Irreversible dynamos in butterflies. In *Proceedings of the 6th International Colloquium on Structural Information & Communication Complexity*, pages 204–218, 1999.
- [60] F. Luccio, L. Pagli, and N. Santoro. Network decontamination in presence of local immunity. *International Journal of Foundations of Computer Science*, 18(3):457–474, 2007.
- [61] A. Montanari and A. Saberi. Convergence to equilibrium in local interaction games. In *Proceedings of the 50th Annual IEEE Symposium on Foundations of Computer Science*, pages 303–312, 2009.
- [62] G. Moran. Parametrization for stationary patterns of the r -majority operators on 0-1 sequences. *Discrete Mathematics*, 132(1–3):175–195, 1994.
- [63] G. Moran. The r -majority vote action on 0-1 sequences. *Discrete Mathematics*, 132(1–3):145–174, 1994.
- [64] G. Moran. On the period-two property of the majority operator in infinite graphs. *Transactions of the American Mathematical Society*, 347(5):1649–1667, 1995.
- [65] S. Morris. Contagion. *Review of Economic Studies*, 67(1):57–78, 2000.
- [66] E. Mossel and S. Roch. On the submodularity of influence in social networks. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing*, pages 128–134, 2007.
- [67] R. Motwani and P. Raghavan. *Randomized Algorithms*. Cambridge University Press, 1995.

- [68] R. Motwani and Y. Xu. Evolution of page popularity under random Web graph models. In *Proceedings of the 25th ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems*, pages 134–142, 2006.
- [69] N. H. Mustafa and A. Pekeč. Majority consensus and the local majority rule. In *Proceedings of the 28th International Colloquium on Automata, Languages and Programming*, pages 530–542, 2001.
- [70] M. E. J. Newman, S. H. Strogatz, and D. J. Watts. Random graphs with arbitrary degree distributions and their applications. *Physical Review E*, 64(2):026118, 2001.
- [71] G. Pandurangan, P. Raghavan, and E. Upfal. Using PageRank to characterize Web structure. *Internet Mathematics*, 3(1):1–20, 2006.
- [72] D. Peleg. Graph immunity against local influence. Technical Report CS96-11, Weizmann Science Press of Israel, 1996.
- [73] D. Peleg. Size bounds for dynamic monopolies. *Discrete Applied Mathematics*, 86(2–3):263–273, 1998.
- [74] D. Peleg. Local majorities, coalitions and monopolies in graphs: A review. *Theoretical Computer Science*, 282(2):231–257, 2002.
- [75] D. M. Pennock, G. W. Flake, S. Lawrence, E. J. Glover, and C. L. Giles. Winners don’t take all: Characterizing the competition for links on the Web. *Proceedings of the National Academy of Sciences*, 99(8):5207–5211, 2002.
- [76] D. A. Pike and Y. Zou. Decycling Cartesian products of two cycles. *SIAM Journal on Discrete Mathematics*, 19(3):651–663, 2005.
- [77] S. Poljak and M. Sura. On periodical behavior in societies with symmetric influences. *Combinatorica*, 3(1):119–121, 1983.
- [78] S. Poljak and D. Turzik. On an application of convexity to discrete systems. *Discrete Applied Mathematics*, 13(1):27–32, 1986.
- [79] M. Richardson and P. Domingos. Mining knowledge-sharing sites for viral marketing. In *Proceedings of the 8th ACM SIGKDD International*

- Conference on Knowledge Discovery and Data Mining*, pages 61–70, 2002.
- [80] B. Samuelsson and J. E. S. Socolar. Exhaustive percolation on random networks. *Physical Review E*, 74(3):036113, 2006.
 - [81] D. Stauffer and A. Aharony. *Introduction to Percolation Theory*. Taylor & Francis, 2nd edition, 1994.
 - [82] M. Steyvers and J. B. Tenenbaum. The large-scale structure of semantic networks: Statistical analyses and a model of semantic growth. *Cognitive Science*, 29(1):41–78, 2005.
 - [83] D. J. Watts. A simple model of global cascades on random networks. *Proceedings of the National Academy of Sciences*, 99(9):5766–5771, 2002.
 - [84] D. B. West. *Introduction to Graph Theory*. Prentice-Hall, 2nd edition, 2001.
 - [85] H. P. Young. *Individual Strategy and Social Structure*. Princeton University Press, Princeton, 1998.
 - [86] H. P. Young. The diffusion of innovations in social networks. In L. E. Blume and S. N. Durlauf, editors, *Economy as an Evolving Complex System*, volume 3 of *Proceedings volume in the Santa Fe Institute studies in the sciences of complexity*, pages 267–282. Oxford University Press, New York, 2006.